Isogeometric analysis for composite sandwich thin plates

Thai Hoang Chien¹, Tran Vinh Loc¹, Nguyen Thoi Trung¹,² and Nguyen Xuan Hung¹,²

Abstract

In this paper, we describe and analyze the application of an isogeometric analysis to static calculation of composite sandwich thin plates. This method is built from Non-Uniform Rational B-spline (NURBS) for both geometry and displacement. Only deflection variables (without rotational degrees of freedom (dof)) are used for each control point. Essential displacements and rotations boundary conditions can be satisfied strongly by assigning control variable values on the boundary and these adjacent to the boundary, respectively. Several numerical examples are illustrated to demonstrate the performance of the present method in comparison with other published methods.

Keywords: NURBS; Isogeometric analysis; composite sandwich plate; thin plate; rotation free.

1. INTRODUCTION

Sandwich structures have been widely used in various engineering such as aircrafts, aerospace, vehicles, buildings, etc. Sandwich structures are made of three layers (two face sheets and a core) with different materials stacked together to achieve desired properties (e.g. high stiffness and strength-to-weight ratios, long fatigue life, wear resistance, lightweight, etc). For the analysis of sandwich plate, the exact elasticity solution first has been proposed by Papano [1] to predict accurately of static behavior. Elasticity solution three-dimensional (3D) can become very expensive when the complex structures are modeled. Generally, computational costs are reduced when two-dimensional model is used. Using two-dimensional model, several plate theories using equivalent single layer have been developed to analyze laminated composite sandwich plates. The classical laminate plate theory (CLPT) [3] can only give good results to thin plates because it ignores the transverse shear deformation. The first-order shear deformation theory (FSDT) [2] can be applied for both moderately thick and thin plates. This theory assumes that transverse shear stresses are constant through the thickness and a shear correction factor is needed to take into account the non-linear distribution of shear stresses. To bypass the limitations of the FSDT, the higher-order shear deformation theories (HSDT) have been developed by Kant et al. [2] for the static analysis of composite sandwich plates based on Navier’s technique. Next, Tran et al. [6] also based on HSDT gave the finite element modeling for bending and vibration analysis of composite sandwich plates. In addition, two-dimensional model based on zigzag theory is also used to calculate the composite sandwich such as: the static analysis of composite sandwich plate with soft-core by Pandit et al.[4]. C⁰ finite element model for the analysis of sandwich laminates with general layup by Singh et al.[5] and an improved C⁰ finite element model for the analysis of laminated sandwich plate with soft-core by Chalak et al.[8], etc.

In the traditional FE method, a discretized geometry obtained through the so-called meshing process is required. This process often leads to geometrical errors even using the higher-order FEM. Also, the communication of the geometry model and the mesh generation during an analysis process that aims to provide the desired accuracy for the solution is always needed and this constitute a time-consuming part in the overall analysis-design process, especially for industrial problems [9]. To overcome this disadvantage, Hughes et al. [9] have recently proposed a NURBS-based isogeometric analysis to bridge the gap between Computer Aided Design (CAD) and Finite Element Analysis (FEA). In contrast to the standard FEM with Lagrange polynomial basis, isogeometric approach utilized more general basis functions such as Non-Uniform Rational B-splines (NURBS) that are common in CAD approaches. Isogeometric analysis is thus very promising because it can directly use CAD data to describe both exact geometry and approximate solution. For structural mechanics, isogeometric analysis has been extensively studied for structural vibrations [10], the Reissner-Mindlin composite plate [13], the composite plate based on HSDT [14], the Reissner-Mindlin shell [11] and Kirchhoff-Love shell [12], etc. The plates are commonly employed in engineering applications as thin plates. So, CLPT is utilized in this paper to reduce computational costs. We focus on NURBS elements using a rotation-free isogeometric formulation for static analysis of composite sandwich plates.

The paper is arranged as follows: a brief of the B-spline and NURBS surface is described in section 2. Section 3 describes an isogeometric approximation for composite sandwich plates. Several numerical examples are illustrated in section 4. Finally we close our paper with some concluding remarks.

2. NURBS-BASED ISOGEOMETRIC ANALYSIS FUNDAMENTALS

2.1. Knot Vectors and Basis Functions

Let be a nondecreasing sequence \(\xi_1, \xi_2, \ldots, \xi_{n+p+1}\) of parameter values, \(\xi_i < \xi_{i+1}, i = 1, \ldots, n + p\). The \(\xi_i\) is called
knots, and $\Xi$ is the set of coordinates in the parametric space. If all knots are equally spaced the knot vector is called uniform. If the first and the last knots are repeated $p + 1$ times, the knot vector is described as open. A B-Spline basis function is $C^\infty$ continuous inside a knot span and $C^{p-1}$ continuous at a single knot. A knot value can appear more than once and is then called a multiple knot. At a knot of multiplicity $k$ the continuity is $C^{p-k}$.

Given a knot vector, the B-spline basis functions $N_{p,i}(\xi)$ of order $p = 0$ are defined recursively on the corresponding knot vector as follows

$$N_{0,i}(\xi) = \begin{cases} 1 & \text{if } \xi_i \leq \xi < \xi_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

(1)

The basis functions of $p > 1$ are defined by the following recursion formula

$$N_{p,i}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{p-1,i}(\xi) + \frac{\xi_{i+p} - \xi}{\xi_{i+p} - \xi_i} N_{p-1,i+p-1}(\xi)$$

(2)

For $p = 0$ and $1$ the basis functions of isogeometric analysis are identical to those of standard piecewise linear and constant finite elements, respectively. However, for $p \geq 2$ they are basically different. In this study, we consider basis functions with $p \geq 2$.

2.2. NURBS Surface

The B-spline curve is defined as:

$$C(\xi) = \sum_{i=1}^{n} N_{p,i}(\xi) P_i$$

(3)

where $P_i$ are the control points and $N_{p,i}(\xi)$ is the $p$th-degree B-spline basis function defined on the open knot vector. Fig. 1 illustrates a set of cubic B-splines curves and cubic B-spline surface for open uniform knot vectors $\Xi = \{0, 0.25, 0.5, 0.75, 1\}$.

The B-spline surfaces are defined by the tensor product of basis functions in two parametric dimensions $\xi$ and $\eta$ with two knot vectors $\Xi = \{\xi_1, \xi_2, ..., \xi_{n+p+1}\}$ and $\eta = \{\eta_1, \eta_2, ..., \eta_{m+q+1}\}$ are expressed as follows:

$$S(\xi, \eta) = \sum_{i,j=1}^{n,m} N_{i,p}(\xi) M_{j,q}(\eta) P_{i,j}$$

(4)

where $P_{i,j}$ is the bidirectional control net, $N_{i,p}(\xi)$ and $M_{j,q}(\eta)$ are the B-spline basis functions defined on the knot vectors over an $n \times m$ net of control points $P_{i,j}$. Similarly to notations used in finite elements, we identify the logical coordinates $(i, j)$ of the B-spline surface with the traditional notation of a node $A$ [11]. Eq.(4) can be rewritten in the following form:

$$S(\xi, \eta) = \sum_{A} N_{A}(\xi, \eta) P_{A}$$

(5)

where $N_{A}(\xi, \eta) = N_{i,p}(\xi) M_{j,q}(\eta)$ is the shape function associated with node $A$.

Similar to B-Splines, a NURBS surface is defined as

$$S(\xi, \eta) = \sum_{A} R_{A}^{\mathbf{w}}(\xi, \eta) P_{A}$$

where $R_{A}^{\mathbf{w}} = \frac{N_{A}^{\mathbf{w}}}{\sum_{A} N_{A}^{\mathbf{w}}}$

(6)

where $w_A$ is the weight function.

3. AN ISOGEOMETRIC FORMULATION FOR THIN PLATE MODEL

Let $\Omega$ be the domain in $R^2$ occupied by the mid-plane of the plate and $u$, $v$, and $w$ denote the displacement components in the $x$, $y$, and $z$ directions, respectively. Using the Kirchhoff model, the displacements of any point in the plate can be expressed as

$$\begin{align*}
u(x, y, z) &= u_0(x, y) - z\theta_x(x, y) \\
v(x, y, z) &= v_0(x, y) - z\theta_y(x, y) \\
w(x, y, z) &= w(x, y)
\end{align*}$$

(7)

where

$$\begin{align*}
\theta_x &= \frac{\partial u}{\partial x} \\
\theta_y &= \frac{\partial v}{\partial y}
\end{align*}$$

(8)

In-plane strains through the following equation:

$$\varepsilon = [\varepsilon_{xx}, \varepsilon_{yy}, \gamma_{xy}] = \varepsilon_0 + \kappa z$$

(9)

where $\varepsilon_0$ and $\kappa$ are the in-plane deformations and curvatures of the middle surface, respectively.

Using the same NURBS basis functions, both the description of the geometry (or the physical point) and the displacement field are expressed as

$$\mathbf{u} = \left[ u_0, v_0, w \right]^T$$

(10)

and

$$\kappa = \left[ 0, 0, -2\frac{\partial^2 w}{\partial x^2 \partial y} \right]^T$$

(11)

The Hook’s law for an arbitrary layer $k$, the stress in plane is expressed as

$$\sigma = \begin{bmatrix} \sigma_x & \sigma_y & \sigma_{xy} \end{bmatrix} = Q \varepsilon$$

(12)

where $\varepsilon$ and $w$ are the strains and the deflection and the material matrix $Q$.

(13)

(14)
functions and \( q_d = [u_d, v_d, w_d] \) is the degrees of freedom of \( u_h \) associated to control point \( A \).

The strains in Eq. (13) can be expressed to following nodal displacements as:

\[
\left[ \varepsilon_x, \varepsilon_y \right] = \sum_{A=1}^{A_{uvw}} \left[ B^m_A, B^b_A \right]^T q_d
\]

(15)

where

\[
B^m_A = \begin{bmatrix} R_{a,x} & 0 & 0 \\ 0 & R_{a,y} & 0 \\ R_{a,x} & R_{a,y} & 0 \end{bmatrix} \quad \text{and} \quad B^b_A = \begin{bmatrix} 0 & 0 & -R_{a,x} \\ 0 & 0 & -R_{a,y} \\ 0 & 0 & -2R_{a,xy} \end{bmatrix}
\]

(16)

\( B^m_A \) and \( B^b_A \) are membrane and bending strain-displacement matrices gained from derivative of shape functions, respectively.

The IGA formulation of composite sandwich plates can then be obtained for static analysis:

\[
Kq = f
\]

(17)

where the global stiffness matrix is

\[
K = \int_{\Omega} \left[ \begin{bmatrix} B^m & A \end{bmatrix} \begin{bmatrix} B^m & B^b \end{bmatrix} \right] d\Omega
\]

(18)

and \( f \) is the global force matrix:

\[
f = \int_{\Omega} p Rd\Omega
\]

where \( q \) are the global displacements vector.

4. NUMERICAL RESULTS

In this section, several numerical studies using an isogeometric analysis are presented. For all numerical examples, quadratic, cubic and quartic NURBS elements integrated with \( nG = (p + 1)(q + 1) \) Gauss points are used.

The material parameters are assumed as:

**Material I:**

\[ E_1 = 25E_2; G_{12} = G_{13} = 0.5E_2; G_{23} = 0.2E_2; \quad v_{12} = 0.25 \]

**Material II:**

Face sheets: \( E_1 = 172.4 \text{ GPa}; \quad E_2 = 6.89 \text{ GPa}; \quad G_{12} = G_{13} = 3.45 \text{ GPa}; \quad G_{23} = 1.378 \text{ GPa}; \quad v_{12} = 0.25 \)

Core: \( E_1 = E_2 = 0.276 \text{ GPa}; \quad G_{12} = 0.1104 \text{ GPa}; \quad G_{23} = 0.414 \text{ GPa}; \quad v_{12} = 0.25 \)

The normalized displacement and in-plane the normal/shear stresses of composite sandwich plate are defined as:

\[
\bar{w} = \frac{10^5wE_2h^2}{q_0a^4}, \quad \bar{\sigma}_x = \frac{\sigma_xh^2}{q_0a^2}, \quad \bar{\sigma}_y = \frac{\sigma_yh^2}{q_0a^2}
\]

and \( \bar{\sigma}_{xy} = \frac{\sigma_{xy}h^2}{q_0a^2} \).

4.1 Three layer \((0^\circ/90^\circ/0^\circ)\) square laminated plate subjected to a sinusoidal load

Let us consider a simply supported square laminated plate subjected to a sinusoidal load \( q = q_0 \sin \left( \frac{\pi x}{a} \right) \sin \left( \frac{\pi y}{b} \right) \).

The length to width ratios is \( a/b=1 \) and the length to thickness ratios is \( a/h=100 \). Material I described previously is used in the numerical calculation. The plate is modeled by \( 9x9, 13x13, 17x17 \) and \( 21x21 \) elements. The convergence of normalized displacement and in-plane stresses are given in Figure 1. It can be seen that, the obtained results are very closed with analytical solutions by Kant [2] based on the third shear deformation plate theory and the elasticity solution 3D by Pagano [1]. The numerical results of normalized transverse displacement and in-plane stresses are given in Table 1. Obtained results are compared with the several other methods including the finite element method (FEM) based on the exponential shear deformation plate theory (ESDT) by Aydogdu [7], the elasticity solution given in Pagano [1] and analytical solutions based on Navier’s technique by Kant [2]. In [2], there are three-solutions such as: the fully third shear deformation plate theory using 12 dof/node (Kant 1), the third shear deformation plate theory proposed by Reddy using 5 dof/node (Kant 2) and the first shear deformation plate theory 5 dof/node (Kant 3). It is observed that for deflection and stresses the results of the NURBS-based method agrees well with published results. Figure 2 plots the distribution of stresses through the thickness of the plate. The obtained results are in good agreement with those reported by Kant [2].

1.2 The sandwich \((0^\circ/\text{core}/0^\circ)\) square plate under sinusoidally distributed load

Table 1. The normalized displacement and the stresses in a three-layer \((0^\circ/90^\circ/0^\circ)\) simply supported square laminate under sinusoidally transverse load

<table>
<thead>
<tr>
<th>Authors &amp; methods</th>
<th>( \bar{w}_{\frac{a}{2}, \frac{b}{2}, 0} )</th>
<th>( \bar{\sigma}_x^{\frac{a}{2}, \frac{b}{2}, \frac{h}{2}} )</th>
<th>( \bar{\sigma}_y^{\frac{a}{2}, \frac{b}{2}, \frac{h}{2}} )</th>
<th>( \bar{\sigma}_{xy}^{(0,0,\frac{h}{2})} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kant 1 [2] (HSDT)</td>
<td>0.4343</td>
<td>0.5392</td>
<td>0.1807</td>
<td>0.0214</td>
</tr>
<tr>
<td>Kant 2 [2] (HSDT)</td>
<td>0.4342</td>
<td>0.5390</td>
<td>0.1806</td>
<td>0.0214</td>
</tr>
<tr>
<td>Aydogdu [7] (ESDT)</td>
<td>0.4350</td>
<td>0.5389</td>
<td>0.1806</td>
<td>0.0214</td>
</tr>
<tr>
<td>Kant 3 [2] (FSDT)</td>
<td>0.4337</td>
<td>0.5384</td>
<td>0.1804</td>
<td>0.0213</td>
</tr>
<tr>
<td>Elasticity [1]</td>
<td>-</td>
<td>0.5390</td>
<td>0.1810</td>
<td>0.0213</td>
</tr>
<tr>
<td>Quadratic (CLPT)</td>
<td>0.4329</td>
<td>0.5383</td>
<td>0.1794</td>
<td>0.0213</td>
</tr>
<tr>
<td>Cubic (CLPT)</td>
<td>0.4342</td>
<td>0.5382</td>
<td>0.1794</td>
<td>0.0213</td>
</tr>
<tr>
<td>Quartic (CLPT)</td>
<td>0.4353</td>
<td>0.5387</td>
<td>0.1796</td>
<td>0.0213</td>
</tr>
</tbody>
</table>
We consider the sandwich \((0^\circ/\text{core}/0^\circ)\) simply supported square plate subjected to sinusoidally distributed load with the thickness of each face sheet equal \(h/10\). Material II is used in this example. The normalized transverse displacement and normalized stresses are reported in Table 2.

The obtained results are compared with the exact elasticity solution by [1], the analytical solution by [2], FEM solutions based on the higher order zigzag plate theory (HOZT) by [4, 5] and and FEM solutions based on the third shear deformation plate theory by [6]. It is found that the results of present method show good agreements with those solutions. The distribution of stresses through the thickness of the plate is illustrated in Figure 3.

1.3 Anti-symmetry the sandwich \((0^\circ/90^\circ/\text{core}/0^\circ/90^\circ)\) square plate under sinusoidal load

In order to study the stretching-bending coupling effect, the anti-symmetry five-layer sandwich plate \((0^\circ/90^\circ/\text{core}/0^\circ/90^\circ)\) is considered. Material II is also used in this problem. The core has a thickness of \(0.8h\) while the two laminated face-sheets are of \(0.1h\).

### Table 2. The normalized displacement and stresses in a three-layer \((0^\circ/\text{core}/0^\circ)\) simply supported square sandwich under sinusoidal transverse load

<table>
<thead>
<tr>
<th>Author &amp; method</th>
<th>(\bar{w}(a, b, 0))</th>
<th>(\bar{\sigma}_x(a, b, h/2))</th>
<th>(\bar{\sigma}_y(a, b, h/3))</th>
<th>(\bar{\sigma}_w(0, 0, h/2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kant 1 [2]</td>
<td>0.8913</td>
<td>1.0990</td>
<td>0.0560</td>
<td>0.0436</td>
</tr>
<tr>
<td>Kant 2 [2]</td>
<td>0.8908</td>
<td>1.0973</td>
<td>0.0549</td>
<td>0.0436</td>
</tr>
<tr>
<td>Kant 3 [2]</td>
<td>0.8852</td>
<td>1.0964</td>
<td>0.0546</td>
<td>0.0435</td>
</tr>
<tr>
<td>Elasticity [1]</td>
<td>-</td>
<td>1.0980</td>
<td>0.0550</td>
<td>0.0437</td>
</tr>
<tr>
<td>Singh et al. [5]</td>
<td>0.9017</td>
<td>1.1020</td>
<td>-</td>
<td>0.0453</td>
</tr>
<tr>
<td>Pandit et al. [4]</td>
<td>0.8917</td>
<td>1.1093</td>
<td>0.0547</td>
<td>0.0434</td>
</tr>
<tr>
<td>Tran et al. [6]</td>
<td>0.8919</td>
<td>1.1069</td>
<td>0.0573</td>
<td>0.0432</td>
</tr>
<tr>
<td>Quadratic</td>
<td>0.8762</td>
<td>1.0894</td>
<td>0.0539</td>
<td>0.0432</td>
</tr>
<tr>
<td>Cubic</td>
<td>0.8800</td>
<td>1.0892</td>
<td>0.0539</td>
<td>0.0431</td>
</tr>
<tr>
<td>Quartic</td>
<td>0.8810</td>
<td>1.0903</td>
<td>0.0540</td>
<td>0.0431</td>
</tr>
</tbody>
</table>

The plate has supported (S) and clamped (C) boundary conditions. The normalized displacement and stresses derived from the present method of a five-layer sandwich plate with various boundary conditions are given in Table 3. For comparison, other methods based on C\(^0\) higher order zigzag plate theory by Chalak et al. [8], Singh et al. [5] and Pandit et al. [4] are cited. It is observed that the present results are in good agreement with published ones for both SCSC and CCCC boundary conditions.

### Table 3. The normalized displacement and stresses in a five-layer \((0^\circ/90^\circ/\text{core}/0^\circ/90^\circ)\) SCSC and CCCC square sandwich under sinusoidal transverse load

<table>
<thead>
<tr>
<th>Boundary conditions Method</th>
<th>(\bar{w}(a, b, 0))</th>
<th>(\bar{\sigma}_x(a, b, h/2))</th>
<th>(\bar{\sigma}_y(a, b, h/2))</th>
<th>(\bar{\sigma}_w(a, b, h/2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>SCSC</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pandit et al. [4]</td>
<td>0.3453</td>
<td>0.4077</td>
<td>0.0326</td>
<td></td>
</tr>
<tr>
<td>Singh et al. [5]</td>
<td>0.3920</td>
<td>0.5986</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Chalak et al. [8]</td>
<td>0.3430</td>
<td>0.4250</td>
<td>0.0366</td>
<td></td>
</tr>
<tr>
<td>Quadratic</td>
<td>0.3177</td>
<td>0.3911</td>
<td>0.0326</td>
<td></td>
</tr>
<tr>
<td>Cubic</td>
<td>0.3318</td>
<td>0.3985</td>
<td>0.0325</td>
<td></td>
</tr>
<tr>
<td>Quartic</td>
<td>0.3323</td>
<td>0.4000</td>
<td>0.0328</td>
<td></td>
</tr>
<tr>
<td>CCCC</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pandit et al. [4]</td>
<td>0.2286</td>
<td>0.4270</td>
<td>0.0228</td>
<td></td>
</tr>
<tr>
<td>Singh et al. [5]</td>
<td>0.2260</td>
<td>0.4283</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Chalak et al. [8]</td>
<td>0.2267</td>
<td>0.4371</td>
<td>0.0259</td>
<td></td>
</tr>
<tr>
<td>Quadratic</td>
<td>0.2106</td>
<td>0.4293</td>
<td>0.0228</td>
<td></td>
</tr>
<tr>
<td>Cubic</td>
<td>0.2212</td>
<td>0.4273</td>
<td>0.0228</td>
<td></td>
</tr>
<tr>
<td>Quartic</td>
<td>0.2216</td>
<td>0.4312</td>
<td>0.0230</td>
<td></td>
</tr>
</tbody>
</table>
5. CONCLUSION

An isogeometric formulation has been developed for static analysis of the composite sandwich thin plates. The present method only used three degrees of freedom per node (3 dof/node), and the obtained results are in very good agreement with analytical solution by Kant 1 [2] using 12 dof/node, Kant 2 [2] using 5 dof/node, FEM solutions using 11 dof/node [4, 5, 8] and FEM solutions using 9 dof/node [6]. The distribution of stresses through the thickness of the sandwich plates are in very good agreement with those of other existing methods.

REFERENCES